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Nonstandard Variational Calculus with Applications to Classical Mechanics. 2. The Inverse Problem and More

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In this paper we continue analyzing the possible applications of nonstandard analysis to variational problems, with particular interest in classical mechanics. In particular, we adapt various techniques of numerical analysis to solve the nonstandard version of the Euler–Lagrange equation for both one- and multidimensional systems. We also start an introductory analysis of the inverse problem of the calculus of variation, identifying a class of nonstandard difference equations for which a first-order Lagrangian can be obtained.

1. INTRODUCTION

In a previous paper[(1)] we discussed the possibility of using nonstandard analysis (NSA)[(2,3)] to formulate, from the very beginning, the problem of the calculus of variation. In other words, we used a complete nonstandard approach to compute the extremum of a given functional $J[y] \equiv \int_a^b F(x, y, y')\,dx$ satisfying the boundary conditions $y(a) = A$ and $y(b) = B$. We showed how this can be done. The action principle produce a set of *ns-finite* algebraic equations whose solution, at least for a large class of models, differs from the standard one, that is, the one obtained by solving the usual Euler–Lagrange differential equation related to the functional $J[q]$, for an infinitesimal quantity. This result was obtained without finding any solution for the nonstandard equations, but only by using estimates of the differences between the standard and the nonstandard solutions. The main result in ref. 1 is, in our opinion, the criterion for the existence and the uniqueness of the solution of a differential equation which no longer makes reference to any

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Lipschitz condition, but only to the estimate of the lower bound of the determinant of a typically tridiagonal matrix. What was lacking in ref. 1 is, of course, a technique for finding the explicit solution of a given nonstandard difference equation. This is the first problem we will solve in this paper, at least for a class of equations. We will show that many suggestions coming from numerical analysis can be easily adapted here, and, due to the nature of NSA, they allow a different way to approach the original problem without any approximation. In particular we will consider some of the examples discussed and not solved in ref. 1.

In the second part we will show how to find the extremum of a functional J depending on n variables,

$$J[q_1, q_2, \ldots, q_n] \equiv \int_{t_i}^{t_f} L(t, q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)\, dt$$

and with given boundary conditions. Many examples of this problem will be discussed in details.

In the last section we will discuss some results on the inverse problem of the calculus of variation. In particular, given some peculiar class of nonstandard difference equations, we will obtain the first-order Lagrangian from which these equations can be derived.

This paper also contains two appendixes which give information about the finite difference equations and the standard inverse problem, in order to keep the paper self-contained.

2. SOLVING THE NONSTANDARD EQUATIONS

In ref.1 we discussed the nonstandard version of the Euler–Lagrange equation for a functional

$$J[q] \equiv \int_{t_i}^{t_f} L(t, q, \dot{q})\, dt \tag{2.1}$$

where $L(t, q, \dot{q})$ is a function with all the first and second partial derivatives continuous and the function $q(t)$ is such that $q(t_i) = q_i$ and $q(t_f) = q_f$. In particular we found that the equation

$$L_q - \frac{d}{dt} L_{\dot{q}} = 0 \tag{2.2}$$

in a nonstandard language must be replaced by the following set of $\Delta - 1$ equations:

$$st\left[\frac{1}{\tau}\left(L\left(t_{k-1}, q_{k-1}, \frac{q_k + \tau - q_{k-1}}{\eta}\right) + L\left(t_k, q_k + \tau, \frac{q_{k+1} - (q_k + \tau)}{\eta}\right)\right.\right.$$
$$\left.\left. - L\left(t_{k-1}, q_{k-1}, \frac{q_k - q_{k-1}}{\eta}\right) - L\left(t_k, q_k, \frac{q_{k+1} - q_k}{\eta}\right)\right)\right] = 0,$$
$$k = 1, 2, \ldots, \Delta - 1 \tag{2.3}$$

We recall that our recipe consists in taking the standard part above, at a first step, with respect to τ and only at a second step also with respect to η.

In ref.1 we focused on some existence results following from the set (2.3). In particular we showed that it is possible to deduce the existence of a unique solution of the variational problem simply by computing, or estimating, the determinant of a tridiagonal matrix. We also discussed many classes of examples in which the solution of the system (2.3) differs from the standard solution for an infinitesimal quantity. Nevertheless we have not yet discussed how to solve the system. In this section we consider in more detail the finite-difference nature of (2.3), applying to this problem some techniques of this branch of mathematics. We refer to Appendix A for information related to the standard difference equations and to some methods of solution.

Let us define the shift operator (of step η) E_η as

$$E_\eta f(x) \equiv f(x + \eta) \tag{2.4}$$

and the difference operator D_η as

$$D_\eta f(x) \equiv f(x + \eta) - f(x) \tag{2.5}$$

so that $D_\eta = E_\eta - 1$. The function $f(x)$ above belongs to a class of opportunely regular functions (for instance, differentiable).

In order to show the details of our procedure we consider a first-order homogeneous differential equation with constant coefficients:

$$\dot{q}(t) + aq(t) = 0 \qquad \text{with} \quad q(t_i) = q_i$$

Of course this equation cannot be produced by a variational problem, but since the procedure of discretization is independent of the degree of the equation, it is still a good starting example.

The standard solution is $q_s(t) = q_i e^{a(t_i - t)}$. From Appendix A and from ref.1 we deduce that the related nonstandard equation can be obtained by replacing the time derivative dq/dt with $D_\eta q/\eta$. Therefore we get

$$(E_\eta + a\eta - 1)q(t) = 0 \qquad \text{with} \quad q(t_i) = q_i$$

Let us observe that this equation gives the whole set (2.3) when η is replaced by $k\eta$, with $k = 1, 2, \ldots, \Delta - 1$. We are going to show that, whenever we want to find explicitly the solution, it is sufficient to fix $k = 1$ and to treat the equation like a difference equation. In particular, using the results of Appendix A, we deduce that the solution of the equation above is $q_{ns}(t) = q_i(1 - a\eta)^{(t_i - t)/\eta}$. Incidentally we observe that, since $st[(1 - a\eta)^{x/\eta}] = e^{-ax}$ for all real x, then the two solutions above belong to the same monad for all $t \in \mathbf{R}$, as we expected.

Before considering more relevant examples, we discuss again the equivalence between the standard and the nonstandard approaches. We have already touched on this point in ref. 1. Now we would like to perform a deeper analysis. The point is the following: let $y(x)$ be the (unique) solution of a given differential equation satisfying certain boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ and let $\{Y_j\}$ be the (unique) solution of the related nonstandard difference equation with $Y_0 = \alpha$ and $Y_\Delta = \beta$. There is no reason *a priori* to be sure that $st[y(x_j) - Y_j] = 0$ for all $j = 0, 1, \ldots, \Delta$, even if the nonstandard difference equation "converges" to the standard differential equation when the standard part is considered. In the case in which this condition is satisfied we will say that the solutions converge. This peculiarity is widely discussed in the literature. For instance, in ref. 4 it is discussed that there exist approximation procedures which do not satisfy the convergence requirement even in the simple case of initial value problems for first-order differential equations. Moreover, to our knowledge, there is no general result about this convergence of the solutions. What can be found in the literature (e.g., refs. 4–6) are only partial results related to explicit models. In these references the problem is always seen from a standard point of view, making no use of nonstandard techniques. Nevertheless most of their results can be translated and used also in our approach.

The most general of the equations treated in the literature[5,6] is $y'' = f(x, y, y')$ with the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ and with the following conditions on f: $0 < Q_* \leq \partial f/\partial y \leq Q^*$ and $|\partial f/\partial y'| \leq P^*$. For this equation, with these hypotheses, in it is shown in ref. 5 that $y(x_j) - Y_j = O(h^2)$, h being the discretization step. This is much more than what is needed by our nonstandard approach. In order to verify the convergence of the solutions it is enough to prove that $y(x_j) - Y_j = O(h^p)$ for some positive real p. Below we discuss this kind of estimate for models which are not of the above form, showing that the solutions "converge" and that therefore the two approaches are equivalent.

We begin with a simple example which already does not fit into the hypotheses above. Let us consider the second-order differential equation with real constant coefficients:

$$\ddot{q} + a\dot{q} + bq = 0, \qquad q(t_i) = q_i, \qquad q(t_f) = q_f \tag{2.6}$$

If $b > 0$, the above condition on $\partial f/\partial y$ is not satisfied, so that we need a different estimate to deduce the convergence of the solutions. The nonstandard version of this equation, obtained as in the previous first-order example, is

$$(E_\eta^2 + E_\eta(a\eta - 2) + (1 - a\eta + b\eta^2))q(t) = 0 \tag{2.7}$$

with the same boundary conditions. The discretization of $\ddot{q}$ is given by $D_\eta^2 q/\eta^2$. In Appendix A we show how the general solution of such an equation can be found. The only difference is that here the discretization step is η and no longer 1. Depending on the values of a and b, equations (2.6) and (2.7) admit different solutions, which must be compared with each other. Let us suppose, for instance, that $a^2 \neq 4b$. Therefore the general solution of the standard equation, once the boundary conditions are implemented, is

$$q_s(t) = c_1^{(s)} e^{\lambda_1 t} + c_2^{(s)} e^{\lambda_2 t}$$

where $\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b})$, $\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$, and

$$c_1^{(s)} = \frac{e^{\lambda_2 t_i} q_f - e^{\lambda_2 t_f} q_i}{e^{\lambda_2 t_i + \lambda_1 t_f} - e^{\lambda_1 t_i + \lambda_2 t_f}}, \qquad c_2^{(s)} = \frac{e^{\lambda_1 t_f} q_i - e^{\lambda_1 t_i} q_f}{e^{\lambda_2 t_i + \lambda_1 t_f} - e^{\lambda_1 t_i + \lambda_2 t_f}}$$

Under the same hypotheses the solution of the nonstandard equation is instead

$$q_{ns}(t) = c_1^{(ns)} q_1^{t/\eta} + c_2^{(ns)} q_2^{t/\eta}$$

where $q_1 = \frac{1}{2}(2 - a\eta + \eta\sqrt{a^2 - 4b})$, $q_2 = \frac{1}{2}(2 - a\eta - \eta\sqrt{a^2 - 4b})$, and

$$c_1^{(ns)} = \frac{q_2^{t_i/\eta} q_f - q_2^{t_f/\eta} q_i}{q_2^{t_i/\eta} q_1^{t_f/\eta} - q_1^{t_i/\eta} q_2^{t_f/\eta}}, \qquad c_2^{(ns)} = \frac{q_1^{t_f/\eta} q_i - q_1^{t_i/\eta} q_f}{q_2^{t_i/\eta} q_1^{t_f/\eta} - q_1^{t_i/\eta} q_2^{t_f/\eta}}$$

What we want to control now is whether the difference $q_s(t) - q_{ns}(t)$ does or does not belong to the monad of zero for all $t \in [t_i, t_f]$. This result is an easy consequence of the equality $st[q_l^{t/\eta}] = e^{\lambda_l t}$ for $l = 1, 2$. We arrive at the same conclusion even for $a^2 = 4b$, so that we can safely conclude that the solution of equation (2.6) is nothing but the standard part of the solution of equation (2.7).

Let us now consider the nonhomogeneous equation related to (2.6),

$$\ddot{q} + a\dot{q} + bq = \Phi(t), \qquad q(t_i) = q_i, \qquad q(t_f) = q_f \tag{2.8}$$

and its nonstandard version

$$(E_\eta^2 + E_\eta(a\eta - 2) + (1 - a\eta + b\eta^2))\, q(t) = \eta^2\Phi(t) \tag{2.9}$$

with the same boundary conditions. Again it is possible to show that the nonstandard solution of (2.9) converges to the standard solution of (2.8), at least for a wide class of functions Φ. To prove this fact, we only need to find a particular solution of the complete equation, since the general solution of the homogeneous equation has already been found. We stress that no substantial difficulty arises in the computation of the coefficients $c_k^{(ns)}$ and $c_k^{(s)}$ when the function Φ is introduced. Of course, this particular solution depends on the form of $\Phi(t)$. We now consider some examples:

(i) Let us suppose that $\Phi(t) = kh^t$. The standard particular solution is $q_s^{(\text{par})}(t) = kh^t/(\log^2 h + a \log h + b)$, while the nonstandard solution is easily obtained, following the suggestions of Appendix A, and has the form

$$q_{ns}^{(\text{par})}(t) = \frac{kh^t}{((h^\eta - 1)/\eta)^2 + a((h^\eta - 1)/\eta) + b}$$

Again, observing that $st[(h^\eta - 1)/\eta] = \log h$, we deduce that $st[q_{ns}^{(\text{par})}(t) - q_s^{(\text{par})}(t)] = 0$ for all $t \in [t_i, t_f]$, so that the general solutions of the complete equations also belong to the same monad.

(ii) Let $\Phi(t)$ be a polynomial of degree n. Then the particular solutions of both (2.8) and (2.9) can be found as a polynomial of the same degree, which certainly exist if $b \neq 0$. These two solutions can be shown to converge to each other using some easy estimates on matrices and their determinants. It is possible in this way to prove that the difference between the two solutions is $O(\eta)$. The same steps can be used even if $\Phi(t)$ is a polynomial times an exponential h^t. Obviously, in this case, the solution also must be sought as a polynomial times the same exponential. We notice that item (i) fits into this more general situation.

(iii) If $\Phi(t)$ is a trigonometric function, eventually multiplied by a factor h^t, the result follows from point (i).

Let us now move to homogeneous equations of higher degree with constant coefficients. We first define two operators

$$L_{(s)} = \sum_{k=0}^{n} a_k \frac{d^{n-k}}{dt^{\,n-k}}, \qquad L_{(ns)} = \sum_{k=0}^{n} a_k \frac{(E_\eta - 1)^{n-k}}{\eta^{n-k}} \tag{2.10}$$

where $a_0 = 1$. With this definition a homogeneous differential equation of degree n can be written as

$$L_{(s)}q(t) = 0 \tag{2.11}$$

while

$$L_{(ns)}q(t) = 0 \tag{2.12}$$

is its nonstandard counterpart. We know that the n independent solutions of the differential equation are obtained by the m ($\leq n$) solutions of its related algebraic equation

$$\sum_{k=0}^{n} a_k \lambda^{n-k} = 0 \tag{2.13}$$

which is the same equation that gives the n independent solutions of the difference equation. We recall that if, for instance, λ_1 is a root of (2.13) with multiplicity one, then $e^{\lambda_1 t}$ and $(1 + \eta\lambda_1)^{t/\eta}$ are respectively solutions of equations (2.11) and (2.12). We observe also that $st[(1 + \eta\lambda_1)^{t/\eta}] = e^{\lambda_1 t}$. Similar conclusions can be obtained if the multiplicity of λ_1 is bigger than one. We can state the following result:

- Given the standard homogeneous differential equation and its nonstandard counterpart in (2.11) and (2.12), there exist a standard fundamental system of solutions $\{q_s^{(j)}(t)\}$ and a corresponding nonstandard fundamental system of solutions $\{q_{ns}^{(j)}(t)\}$, $j = 1, \ldots, n$, such that $st\left|q_s^{(j)}(t) - q_{ns}^{(j)}(t)\right| = 0$ for all $j = 1, \ldots, n$ and for all $t \in [t_i, t_f]$.

Up to this point we have not yet introduced the boundary conditions. If n is even, as surely happens if the equation follows from a variational principle,[7] we can fix the n boundary conditions on $q(t)$ and its successive derivatives up to the order $n/2$. These conditions should be sufficient to fix the coefficients both in $q_s(t) = \Sigma_{i=1}^{n} c_i^{(s)} q_s^{(i)}(t)$ and in $q_{ns}(t) = \Sigma_{i=1}^{n} c_i^{(ns)} q_{ns}^{(i)}(t)$, where $\{q_s^{(i)}\}$ and $\{q_{ns}^{(i)}\}$ are just the fundamental solutions introduced above. In order to conclude that $q_{ns}(t) \approx q_s(t)$ for any time, it suffices to show now that $c_i^{(ns)}(t) \approx c_i^{(s)}(t)$ for all $i = 1, \ldots, n$. This is actually true and can be shown using techniques analogous to the ones used for the homogeneous equations (2.6) and (2.7).

The same results can also be extended to nonhomogeneous equations. The convergence of the solutions is easily proved for nonhomogeneous just as for the ones discussed for a second-order equation, with very similar techniques.

Many other differential equations are under control as far as the convergence of the solutions is concerned. We do not discuss these other examples here since they are related to initial value problems more than to boundary condition problems. Therefore their relation to a variational approach is not so evident. In particular nth-order-differential equations in normal form could be discussed. We plan to come back to this subject in the near future.

We conclude this section by showing how two of the examples considered in ref. 1 can be explicitly solved using the procedure proposed in this section.

The first example is given by the differential equation $\ddot{q}(t) = 1$, with the boundary conditions $q(0) = q(1) = 0$. Its standard solution is $q_s(t) = (t^2 - t)/2$. The nonstandard version of $\ddot{q}(t) = 1$ is obviously $(E_\eta^2 - 2E_\eta + 1)q = \eta^2$. The general solution of the homogeneous equation is $q_{ns}^{(h)}(t) =$

$c_1 + c_2 t$, while $t^2/2$ is a particular solution of the complete equation. Fixing the above boundary conditions, we obtain the solution $q_{ns}(t) = (t^2 - t)/2$. Therefore the standard and nonstandard solutions coincide!

The situation is different for the second example, $\ddot{q} = q + t$, with boundary conditions $q(0) = q(1) = 0$, whose standard solution is $q_s(t) = (e^t - e^{-t})/(e - e^{-1}) - t$. The nonstandard equation is now $(E_\eta^2 - 2E_\eta + (1 - \eta^2))q = t\eta^2$. Using the same boundary conditions, we obtain now

$$q_{ns}(t) = \frac{(1 + \eta)^{t/\eta} - (1 - \eta)^{t/\eta}}{(1 + \eta)^{1/\eta} - (1 - \eta)^{1/\eta}} - t$$

Recalling that $st[(1 \pm \eta)^{t/\eta}] = e^{\pm t}\ \forall t \in \mathbf{R}$, we can easily verify that $st[q_s(t) - q_{ns}(t)] = 0\ \forall t \in \mathbf{R}$.

3. THE EULER–LAGRANGE EQUATIONS IN SEVERAL VARIABLES

In this section we will obtain the analog of the system (2.3) for a functional J depending on more variables.

Let us consider a function $L(t, q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$, with all the first and second partial derivatives continuous. We define the functional

$$J[q_1, q_2, \ldots, q_n] \equiv \int_{t_i}^{t_f} L(t, q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)\, dt \qquad (3.1)$$

The obvious generalization of the variational problem in one dimension consists now in obtaining the set of functions $(q_1(t), q_2(t), \ldots, q_n(t))$, satisfying the following boundary conditions:

$$q_k(t_i) = q_{i,k}, \qquad q_k(t_f) = q_{f,k}, \qquad k = 1, 2, \ldots, n$$

which are extrema of the functional J. In this case it is quite easy to generalize the results of ref. 1. With the very same steps and considerations as in ref. 1 we obtain the following set of n times $\Delta - 1$ equations:

$$st\Bigg[\frac{1}{\tau}\Bigg(L\bigg(t_{k-1}, q_{\alpha,k-1}, \frac{q_{\alpha,k} + \tau - q_{\alpha,k-1}}{\eta}\bigg) + L\bigg(t_k, q_{\alpha,k} + \tau, \frac{q_{\alpha,k+1} - (q_{\alpha,k} + \tau)}{\eta}\bigg) - L\bigg(t_{k-1}, q_{\alpha,k-1}, \frac{q_{\alpha,k} - q_{\alpha,k-1}}{\eta}\bigg) - L\bigg(t_k, q_{\alpha,k}, \frac{q_{\alpha,k+1} - q_{\alpha,k}}{\eta}\bigg)\Bigg)\Bigg] = 0$$

$$k = 1, 2, \ldots, \Delta - 1, \qquad \alpha = 1, 2, \ldots, n \qquad (3.2)$$

The rule for dealing with these equations is the same we already used for the one-dimensional case: we first take the standard part with respect to τ

and only after obtaining the solution do we do this also with respect to η. Of course these equations are the nonstandard analogs of the well-known set of n differential equations

$$L_{q\alpha} - \frac{d}{dt} L_{\dot{q}\alpha} = 0\,, \qquad \alpha = 1, 2, \ldots, n \tag{3.3}$$

We now consider some examples.

Example 1. $L(r(t), \theta(t), \dot{r}(t), \dot{\theta}(t)) = \frac{1}{2}m(\dot{r}(t)^2 + r(t)^2\dot{\theta}(t)^2) - \frac{1}{2}k(r(t) - \bar{r})^2 + mgr(t)\cos\theta(t)$.

This first example describes a particle with mass m free to move in a vertical plane, and fixed to the origin of the plane by means of an elastic string. Therefore this particle is subject to gravity and to the quadratic force of the string expressed by Hooke's law. Here $\bar{r}$ is the equilibrium position of the elastic string. We will not discuss the explicit solution of the system (3.2) since even its standard counterpart is hard to manage. We will focus our attention only on the relations between the standard and the nonstandard approaches. The classical Euler–Lagrange equations are

$$\begin{aligned} m\ddot{r} &= mr\dot{\theta}^2 - k(r - \bar{r}) + mg\cos\theta \\ 2\dot{r}\dot{\theta} + r^2\ddot{\theta} &= -gr\sin\theta \end{aligned}$$

From algebraic calculations we get

$$L\left(r_{k-1}, \theta_{k-1}, \frac{r_k + \tau - r_{k-1}}{\eta}, \frac{\theta_k - \theta_{k-1}}{\eta}\right) - L\left(r_{k-1}, \theta_{k-1}, \frac{r_k - r_{k-1}}{\eta}, \frac{\theta_k - \theta_{k-1}}{\eta}\right)$$
$$= \frac{1}{2} m \left[\frac{\tau^2}{\eta^2} + 2\frac{\tau}{\eta^2}(r_k - r_{k-1})\right]$$

and

$$L\left(r_k + \tau, \theta_k, \frac{r_{k+1} - r_k - \tau}{\eta}, \frac{\theta_{k+1} - \theta_k}{\eta}\right) - L\left(r_k, \theta_k, \frac{r_{k+1} - r_k}{\eta}, \frac{\theta_{k+1} - \theta_k}{\eta}\right)$$
$$= \frac{1}{2} m \left[\frac{\tau^2}{\eta^2} - 2\frac{\tau}{\eta^2}(r_{k+1} - r_k) + (\tau^2 + 2\tau r_k)\left(\frac{\theta_{k+1} - \theta_k}{\eta}\right)^2\right]$$
$$- \frac{1}{2}k\tau^2 - k\tau(r_k - \bar{r}) + mg\tau\cos\theta_k$$

Therefore the first equation can be written as

$$m\,\frac{2r_k - r_{k-1} - r_{k+1}}{\eta^2} + mr_k\left(\frac{\theta_{k+1} - \theta_k}{\eta}\right)^2 - k(r_k - \bar{r}) + mg\cos\theta_k \approx 0$$

With similar calculations we get the second equation in the following form:

$$r_k^2\,\frac{2\theta_k - \theta_{k-1} - \theta_{k+1}}{\eta^2} - 2r_k\,\frac{\theta_{k+1} - \theta_k}{\eta}\,\frac{r_{k+1} - r_k}{\eta} - gr_k\sin\theta_k \approx 0$$

In obtaining this equation we have taken advantage of the fact that, since by hypothesis $r(t)$ is a twice differentiable function, r_k, r_{k-1}, and r_{k+1} all belong to the same monad for all k.

We observe that, as expected, the nonstandard equations are the discretized versions of the standard ones. This is in agreement with Proposition 2 of ref. 1, which also can be restated for this multidimensional situation: without going into details, which are indeed very similar to those in ref. 1, we can say that the system (3.2) is the discretized version of the standard system (3.3). Therefore, we may say that the discretization procedure *commutes* with the procedure of finding the variation of the functional $J[q_1, q_2, \ldots, q_n]$. This means that the two procedures (a) and (b) described below give equations which, at most, differ for infinitesimal quantities:

(a1) Compute δJ and put $\delta J = 0$. This gives the standard Euler–Lagrange equations.
(a2) Discretize the system obtained in point (a1) using an infinitesimal time step η.
(b1) Discretize the functional $J[q_1, q_2, \ldots, q_n]$ by means of an infinitesimal time step η.
(b2) Compute the variation of the discretized functional J, and assume that this quantity is infinitesimal. This gives back the system (3.2).

Example 2. $L(q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t)) = \dot{q}_1^2(t) + \dot{q}_2^2(t) + 2q_1(t) + 2q_2(t)$.

This example is really extremely simple since the Lagrangian is obviously a sum of two one-body contributions and there is no "interaction" between particle 1 and particle 2. Let us fix the following boundary conditions:

$$q_\alpha(0) = q_\alpha(1) = 0\,, \qquad \alpha = 1, 2$$

The standard solution is $q_1(t) = q_2(t) = \frac{1}{2}(t^2 - t)$. We will see in a moment that the same solution is found using the nonstandard equations (3.2).

With some algebraic computation we obtain

$$
\begin{aligned}
&L\left(q_{1,k-1},\, q_{2,k-1},\, \frac{q_{1,k}+\tau-q_{1,k-1}}{\eta},\, \frac{q_{2,k}-q_{2,k-1}}{\eta}\right)\\
&\quad - L\left(q_{1,k-1},\, q_{2,k-1},\, \frac{q_{1,k}-q_{1,k-1}}{\eta},\, \frac{q_{2,k}-q_{2,k-1}}{\eta}\right)\\
&\quad = \frac{\tau^2}{\eta^2} + 2\,\frac{\tau}{\eta^2}\,(q_{1,k}-q_{1,k-1})
\end{aligned}
$$

and

$$
\begin{aligned}
&L\left(q_{1,k}+\tau,\, q_{2,k},\, \frac{q_{1,k+1}-q_{1,k}-\tau}{\eta},\, \frac{q_{2,k+1}-q_{2,k}}{\eta}\right)\\
&\quad - L\left(q_{1,k},\, q_{2,k},\, \frac{q_{1,k+1}-q_{1,k}}{\eta},\, \frac{q_{2,k+1}-q_{2,k}}{\eta}\right)\\
&\quad = \frac{\tau^2}{\eta^2} - 2\,\frac{\tau}{\eta^2}\,(q_{1,k+1}-q_{1,k}) + 2\tau
\end{aligned}
$$

so that, after taking the standard part with respect to τ, the first equation becomes

$$2q_{1,k} - q_{1,k-1} - q_{1,k+1} = -\eta^2, \qquad 1 \le k \le \Delta - 1 \tag{3.4}$$

This coincides with the nonstandard equation found in Example 1 of ref. 1 for the single variable q_k, and it is, as discussed in that reference, the discretized version of the standard differential equation.

Due to the symmetry $q_1 \leftrightarrow q_2$ of the Lagrangian an equation analogous to (3.4) is obtained for $q_2(t)$. The solution of this equation is discussed in the previous section. We see that, considering the boundary conditions above, the solution coincides with the standard one for all $t \in [0, 1]$.

Example 3. $L(q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t)) = \frac{1}{2}m(\dot{q}_1^2(t) + \dot{q}_2^2(t)) - U(q_1(t) - q_2(t))$.

This is actually a family of examples, depending on the explicit form of the potential U. It describes two particles with the same mass "living" in one spatial dimension which interact with each other by means of a force depending only on the mutual distance.

If we call $U_x(x) \equiv dU(x)/dx$, we can write the standard equations as

$$
\begin{aligned}
m\ddot{q}_1 &= -U_x(q_1 - q_2)\\
m\ddot{q}_2 &= U_x(q_1 - q_2)
\end{aligned}
$$

These equations are particularly easy to manage since they can be easily decoupled simply by defining two new independent variables $X^{\pm} \equiv q_1 \pm q_2$. In this way we get

$$\ddot{X}^{+} = 0$$

$$\ddot{X}^{-} = -\frac{2}{m} U_x(X^{-}) \tag{3.5}$$

On the other hand the nonstandard equations obtained by (3.2) are

$$m \frac{2q_{1,k} - q_{1,k-1} - q_{1,k+1}}{\eta^2} \approx U_x(q_{1,k} - q_{2,k})$$

$$m \frac{2q_{2,k} - q_{2,k-1} - q_{2,k+1}}{\eta^2} \approx -U_x(q_{1,k} - q_{2,k})$$

Again it is possible to decouple these equations: let us define $X_k^{\pm} \equiv q_{1,k} \pm q_{2,k}$ for all k. We obtain

$$2X_k^{+} - X_{k-1}^{+} - X_{k+1}^{+} \approx 0$$

$$2X_k^{-} - X_{k-1}^{-} - X_{k+1}^{-} \approx \frac{2\eta^2}{m} U_x(X_k^{-}) \tag{3.6}$$

which are, of course, just the nonstandard version of (3.5).

Until we fix the form of the potential, we cannot say anything more. We now consider two different choices of U and see what happens.

Example 3A. $U(q_1 - q_2) = \frac{1}{2}(q_1 - q_2)^2$.

With this choice of potential the nontrivial equation in (3.5) becomes the equation of a harmonic oscillator $\ddot{X}^{-} + \omega^2 X^{-} = 0$, where $\omega = 2k/m$.

We fix the following boundary conditions: $q_1(0) = q_2(0) = 0$ and $q_1(1) = q_2(1) = 1$. We must use care in fixing the boundary conditions if we want to obtain a unique solution of the variational problem (as usually happens when we deal with harmonic oscillators).

The standard equations for X^{+} and X^{-} are easily solved and, once the boundary conditions above are implemented, we get the unique solution $q_1(t) = q_2(t) = t$.

It may be interesting to observe that the different set of boundary conditions $q_1(0) = q_2(0) = 0$ and $q_1(\pi/\omega) = q_2(\pi/\omega) = 0$ would give instead the infinite set of solutions $q_1(t) = -q_2(t) = D \sin t$, where D is an arbitrary constant.

Let us now consider the nonstandard approach. It can be deduced from the results contained in the previous section that equation (3.6) for the variable X_k^{+} has the solution $X^{+}(t) = A + Bt$, where A and B are arbitrary constants to be fixed by the boundary conditions. In fact, introducing the shift operator, we can rewrite this equation as $(E_\eta^2 - 2E_\eta + 1)X_{k-1}^+ = 0$, which coincides with equation (2.7) with $a = b = 0$. The equation for X_k^-, $2X_k^- - X_{k-1}^- - X_{k+1}^- \approx \omega^2\eta^2 X_k^-$, is solved by $X^-(t) = Cs_1^{t/\eta} + Ds_2^{t/\eta}$, where again C and D are constants and

$$s_{1,2} \equiv \frac{2 - \omega^2\eta^2 \pm \omega\eta\sqrt{\omega^2\eta^2 - 4}}{2}.$$

First of all it is interesting to observe that $st[s_1^{t/\eta}] = e^{i\omega t}$ and $st[s_2^{t/\eta}] = e^{-i\omega t}$, so that the general solution strongly resembles the standard one. Furthermore, once the boundary conditions $q_1(0) = q_2(0) = 0$ and $q_1(1) = q_2(1) = 1$ are fixed, and once the standard part of the complete solution is taken, we obtain back exactly the standard solution.

Example 3B. $U(q_1 - q_2) = -(q_1 - q_2)^3\{1 + (q_1 - q_2)\}$.

This is a second example fitting in the class of Example 3, and again it can be exactly solved.

The first equation in (3.5) gives the usual free solution $X^+(t) = A + Bt$, while the second one, $\ddot{X}^- = 3(X^-)^2 + 4(X^-)^3$, is a bit more difficult and can be solved by introducing a new variable $Z(X) \equiv \dot{X}^-(t)$. We have put $m = 2$ for simplicity of notation. Fixing the initial conditions $q_1(0) = \dot{q}_1(0) = \dot{q}_2(0) = 0$ and $q_2(0) = 1$, we get the solutions $q_1(t) = 1/2 + 1/(t^2 - 2)$ and $q_2(t) = 1/2 - 1/(t^2 - 2)$.

Let us consider the nonstandard approach. For the same $m = 2$ the system (3.6) can be rewritten as

$$2X_k^+ - X_{k-1}^+ - X_{k+1}^+ \approx 0$$

$$2X_k^- - X_{k-1}^- - X_{k+1}^- \approx -\eta^2(3(X_k^-)^2 + 4(X_k^-)^3)$$

The first equation is the usual free particle equation, whose solution is $X^+(t) = A + Bt$, while the second is nonlinear and can be solved by defining a new set of variables $Z_k \equiv (X_{k+1}^- - X_k^-)/\eta$. In these new variables the equation can be rewritten as $(Z_{k+1} - Z_k)Z_k \approx (3(X_k^-)^2 + 4(X_k^-)^3)(X_{k+1}^- - X_k^-)$, whose solution is $\frac{1}{2}Z_k^2 = (X_k^-)^3 + (X_k^-)^4$, as is easily verified. This last equation can be further managed and it gives the solution $X^-(t) = 2/(t^2 - 2)$. Fixing the same initial conditions as in the standard situation, we obtain again the same solutions as before.

4. THE INVERSE PROBLEM: A FIRST APPROACH

We devote this section to a first approach to the inverse problem of the calculus of variation. The standard inverse problem is concerned with the possibility of finding a Lagrangian L which, by means of the Euler–Lagrange equations (3.3), returns a given set of differential equations. In the nonstandard version of this problem we simply replace the set (3.3) with its nonstandard counterpart (3.2). We will not consider the general situation, which is quite difficult solve, but only models for which a solution can be easily found. These models will be chosen also for their physical relevance; indeed they describe classical equations of motion for point particles. We will also obtain a no-go result. We refer to Appendix B for a brief resume of the standard approach to the inverse problem.

Let us consider the following system of first-order nonstandard difference equations:

$$\begin{cases} q_{1,k+1} - q_{1,k} \approx \eta\phi_1(q_{1,k}, q_{2,k}, \ldots, q_{n,k}, t_k) \\ q_{2,k+1} - q_{2,k} \approx \eta\phi_2(q_{1,k}, q_{2,k}, \ldots, q_{n,k}, t_k) \\ \ldots \\ \ldots \\ q_{n,k+1} - q_{n,k} \approx \eta\phi_n(q_{1,k}, q_{2,k}, \ldots, q_{n,k}, t_k) \end{cases} \tag{4.1}$$

which is the nonstandard version of system (B.1) in Appendix B. Here η is an infinitesimal. In Appendix B we discuss some results related to the standard approach. In this section we will discuss the analogs of those results in the nonstandard approach.

First we recall that any nth-order difference equation can be rewritten in this form simply by introducing new variables exactly in the same way as we do for an nth-order differential equation.

We start by discussing the existence of a first-order Lagrangian

$$L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = \sum_{\alpha=1}^{n} \dot{q}_\alpha f_\alpha(q_1, \ldots, q_n, t) + f_0(q_1, \ldots, q_n, t) \tag{4.2}$$

which, together with the system (3.2), returns system (4.1). The details are long and not so interesting and therefore will be omitted. Assuming that all the functions f_β are twice differentiable, it is a straightforward exercise to obtain the nonstandard version of equation (B.4). Defining

$$M^k_{\alpha,\beta} \equiv \frac{f_\alpha(q_{1,k}, \ldots, q_{\beta,k+1}, \ldots, q_{n,k}, t_k) - f_\alpha(q_{1,k}, \ldots, q_{\beta,k}, \ldots, q_{n,k}, t_k)}{q_{\beta,k+1} - q_{\beta,k}} - \frac{f_\beta(q_{1,k}, \ldots, q_{\alpha,k} + \tau, \ldots, q_{n,k}, t_k) - f_\beta(q_{1,k}, \ldots, q_{\alpha,k}, \ldots, q_{n,k}, t_k)}{\tau} \tag{4.3}$$

and

$$F^k_\alpha(q_{1,k}, \ldots, q_{\alpha,k}, \ldots, q_{n,k}, t_k, \tau, \eta)$$
$$\equiv -\frac{f_\alpha(q_{1,k}, \ldots, q_{n,k}, t_{k+1}) - f_\alpha(q_{1,k}, \ldots, q_{n,k}, t_k)}{\eta}$$
$$+ \frac{f_0(q_{1,k}, \ldots, q_{\alpha,k} + \tau, \ldots, q_{n,k}, t_k) - f_0(q_{1,k}, \ldots, q_{\alpha,k}, \ldots, q_{n,k}, t_k)}{\tau} \tag{4.4}$$

then the functions f_β in the Lagrangian must be taken to be any set of solutions of the system

$$\sum_{\beta=1}^{n} M^k_{\alpha\beta}\phi_\beta = F^k_\alpha, \qquad \alpha = 1, 2, \ldots, n, \qquad k = 0, 1, 2, \ldots, \Delta - 1 \tag{4.5}$$

Before considering some examples in which the Lagrangian can be found, we state the announced no-go result. If n is an odd integer, then it is easy to see that the skew-symmetric matrix M^k, with matrix elements $M^k_{\alpha\beta}$, is singular for any k. It is therefore not possible to invert the system (4.5) in order to return to the original system (4.1). The situation changes drastically if n is an even integer, since a skew-symmetric matrix with an even number of rows and columns may have, in general, a nonzero determinant. This result is in agreement with its known standard version: in both cases, for odd n, there exists no first-order Lagrangian solving the inverse problem.

Now we analyze in some detail the nonstandard version of the results discussed in Appendix B for the easiest nontrivial situation: $n = 2$. This is not a useless task since, as is well known, (almost) any equation of motion of a single classical particle can be rewritten in this form. The system we will deal with is therefore

$$\begin{cases} \dfrac{q_{1,k+1} - q_{1,k}}{\eta} \approx \phi_1(q_{1,k}, q_{2,k}) \\ \dfrac{q_{2,k+1} - q_{2,k}}{\eta} \approx \phi_2(q_{1,k}, q_{2,k}) \end{cases} \tag{4.6}$$

where the functions on the right-hand sides are now supposed to be time independent, and the tentative Lagrangian is therefore

$$L(q_1, q_2, \dot q_1, \dot q_2) = \dot q_1 f_1(q_1, q_2) + \dot q_2 f_2(q_1, q_2) + f_0(q_1, q_2) \tag{4.7}$$

Let us notice that, due to time independence of the ϕ_α, even the first-order

Lagrangian can be sought with no explicit dependence on t. The system (4.5) reduces now to the two equations

$$\begin{cases} M_{1,2}^k \phi_2(q_{1,k}, q_{2,k}) \approx F_1^k \\ M_{2,1}^k \phi_1(q_{1,k}, q_{2,k}) \approx F_2^k \end{cases} \tag{4.8}$$

where $M_{1,2}^k \approx -M_{2,1}^k$ and F_α^k are defined in (4.4), but do not depend explicitly on t.

We now consider different classes fixed by particular choices of the functions ϕ_α, the same ones discussed in Appendix B from a standard point of view. We will show that in these situations the system (4.8) can be solved and the functions f_α give rise to the same Lagrangian as in the standard situation.

Condition 1. $\phi_1(q_1, q_2) = \phi_1(q_2)$, $\phi_2(q_1, q_2) = \phi_2(q_1)$.

If we put $f_1(q_1, q_2) = q_2$ and $f_2(q_1, q_2) = 0$, from the definition of $M_{1,2}^k$ it follows that $M_{1,2}^k = 1 = -M_{2,1}^k$, so that the system (4.8) takes the form

$$\begin{cases} 1\phi_2(q_{1,k}) \approx F_1^k \\ -1\phi_1(q_{2,k}) \approx F_2^k \end{cases}$$

From the definition of F_α^k it immediately follows that $f_0(q_1, q_2) = \int \phi_2(q_1)\, dq_1 - \int \phi_1(q_2)\, dq_2$, so the Lagrangian coincides with the one given in Appendix B.

A simple example of this situation is given by the following system:

$$\begin{cases} \dfrac{q_{1,k+1} - q_{1,k}}{\eta} \approx q_{2,k}^2 \\ \dfrac{q_{2,k+1} - q_{2,k}}{\eta} \approx q_{1,k} + 1 \end{cases}$$

Due to the above results, it is immediate to find the Lagrangian for this set of nonstandard equations: $L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \dot{q}_1 q_2 + \frac{1}{2}q_1^2 + q_1 - \frac{1}{3}q_2^3$. We can verify the rightness of this result simply by substituting this Lagrangian in equation (3.2).

Condition 2. $\phi_1(q_1, q_2) = q_2$, $\phi_2(q_1, q_2) = \phi_2(q_1)$.

This is actually a particular situation of the previous case. We prefer to consider this situation separately since it describes the well-known standard equation of motion $\ddot{x} = f(x)$. Of course the solution for the functions f_β directly follows from the results above: $f_1(q_1, q_2) = q_2$, $f_2(q_1, q_2) = 0$, and $f_0(q_1, q_2) = \int \phi_2(q_1)\, dq_1 - q_2^2/2$.

Let us now consider a simple example. We consider the following system:

$$\begin{cases} \dfrac{q_{1,k+1} - q_{1,k}}{\eta} \approx q_{2,k} \\ \dfrac{q_{2,k+1} - q_{2,k}}{\eta} \approx -q_{1,k} \end{cases}$$

This is the nonstandard version of the well-known differential equation $\ddot{x} = -x$. Using the above definition, we find the expression of our Lagrangian: $L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \dot{q}_1 q_2 - \frac{1}{2}(q_1^2 + q_2^2)$. Again, if we use this Lagrangian in (3.2), we obtain the above system.

Condition 3. $\phi_1(q_1, q_2) = q_2$, $\phi_2(q_1, q_2) = \phi_2(q_2)$.

This is a classical equation of motion of the following type: $\ddot{x} = f(\dot{x})$; therefore it describes a one-dimensional particle subjected to a given friction. Again we put $f_2(q_1, q_2) = 0$ and $f_1(q_1, q_2) = -\int dq_2/\phi_2(q_2)$. In this way the system (4.8) takes the form

$$\begin{cases} \dfrac{-1}{\phi_2(q_{2,k})} \phi_2(q_{2,k}) \approx F_1^k \\ \dfrac{1}{\phi_2(q_{2,k})} q_{2,k} \approx F_2^k \end{cases}$$

which is solved by

$$f_0(q_1, q_2) = -q_1 + \int \frac{q_2 \, dq_2}{\phi_2(q_2)}$$

An example of this situation is given by

$$\begin{cases} \dfrac{q_{1,k+1} - q_{1,k}}{\eta} \approx q_{2,k} \\ \dfrac{q_{2,k+1} - q_{2,k}}{\eta} \approx q_{2,k}^2 \end{cases}$$

Using the above results, we deduce the following form for the Lagrangian: $L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \dot{q}_1/q_2 - q_1 + \log|q_2|$. Again, it is a simple exercise to obtain the above system from this Lagrangian by means of equations (3.2).

Condition 4. $\phi_1(q_1, q_2) = q_2$, $\phi_2(q_1, q_2) = \Phi(q_1)\, \Psi(q_2)$.

This situation describes a classical equation of motion of the type $\ddot{x} =$

$\Phi(x)\ \Psi(\dot{x})$. This time it is convenient to put $f_1(q_1, q_2) = 0$ and $f_2(q_1, q_2) = q_1/\Psi(q_2)$. In this way the system (4.8) becomes

$$\begin{cases} \dfrac{-1}{\Psi(q_{2,k})}\,\Phi(q_{1,k})\Psi(q_{2,k}) \approx F_1^k \\[2ex] \dfrac{1}{\Psi(q_{2,k})}\,q_{2,k} \approx F_2^k \end{cases}$$

which is solved by

$$f_0(q_1, q_2) = -\int \Phi(q_1)\,dq_1 + \int \frac{q_2\,dq_2}{\Psi(q_2)}$$

An example is given by the following system of nonstandard equations

$$\begin{cases} \dfrac{q_{1,k+1} - q_{1,k}}{\eta} \approx q_{2,k} \\[2ex] \dfrac{q_{2,k+1} - q_{2,k}}{\eta} \approx q_{1,k}^2\,q_{2,k} \end{cases}$$

In this example the above definitions return $L(q_1, q_2, \dot{q}_1, \dot{q}_2) = (\dot{q}_2/q_2)q_1 + q_2 - q_1^3/3$.

Condition 5. $\phi_1(q_1, q_2) = \phi_1(q_1)$, $\phi_2(q_1, q_2) = \phi_2(q_2)$.

This situation does not describe a second-order equation of motion, but rather two decoupled equations $\dot{q}_1 = \phi_1(q_1)$ and $\dot{q}_2 = \phi_2(q_2)$. Due to the no-go result discussed in this section we know that it is not possible to find a first-order Lagrangian returning this set of equations as a sum of two contributions each depending on only one variable. Nevertheless a solution can be found when q_1 and q_2 are considered, in a sense, as coupled variables. Let us put

$$f_1(q_1, q_2) = \frac{1}{\phi_1(q_1)}\int \frac{dq_2}{\phi_2(q_2)} \qquad \text{and} \qquad f_2\,(q_1, q_2) = 0$$

The system (4.8) now can be rewritten as

$$\begin{cases} \dfrac{1}{\phi_1(q_{1,k})\phi_2(q_{2,k})}\,\phi_2(q_{2,k}) \approx F_1^k \\[2ex] \dfrac{-1}{\phi_1(q_{1,k})\phi_2(q_{2,k})}\,\phi_1(q_{1,k}) \approx F_2^k \end{cases}$$

which is immediately seen to be solved by

$$f_0(q_1, q_2) = \int \frac{dq_1}{\phi_1(q_1)} - \int \frac{dq_2}{\phi_2(q_2)}$$

An example is given by the system

$$\begin{cases} \dfrac{q_{1,k+1} - q_{1,k}}{\eta} \approx q_{1,k} + 1 \\ \dfrac{q_{2,k+1} - q_{2,k}}{\eta} \approx \dfrac{1}{q_{2,k}} \end{cases}$$

The Lagrangian is now

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{\dot{q}_1 q_2^2}{2(q_1 + 1)} + \log\left|q_1 + 1\right| - \frac{q_2^2}{2}$$

All the above integrals, which are assumed to exist, must be understood in the nonstandard sense.

We have considered here five classes of difference equations for which it is simple to find a first-order Lagrangian. Different classes can also be treated; we will consider these generalizations in a future paper.

We also wish to use NSA to discuss the variational principle in quantum mechanics:

$$\delta\langle\Psi, H\Psi\rangle = 0$$

where Ψ is the wave function and H is the hamiltonian of the system. We expect to obtain, with some appropriate procedure, the nonstandard analog of the Schrödinger equation. Further generalizations will hopefully produce also the nonstandard relativistic (Dirac or Klein–Gordon) equations of motion.

APPENDIX A. SOLUTIONS OF SOME STANDARD DIFFERENCE EQUATIONS

In this appendix, included here for completeness, we will briefly summarize some techniques of solution of some particular difference equations. We refer to ref. 8 for a simple but detailed analysis.

Let E be the shift operator acting on regular functions in the canonical way: $Eu(x) \equiv u(x + 1)$.

For a general homogeneous linear equation of order n with real constant coefficients of the form

$$L(E)u(x) \equiv (b_0E^n + b_1E^{n-1} + \ldots + b_n)u(x) = 0 \qquad \text{(A.1)}$$

the solution must be sought in the form $u(x) = Ce^{mx}$, where C is a constant and m is fixed by the equation. Calling $q \equiv e^m$, the equation becomes $b_0q^n + b_1q^{n-1} + \ldots + b_n = 0$, which gives three different situations, depending on the solutions q_α of the equation:

(i) The n roots are all different. In this case the general solution of equation (A.1) is $u(x) = C_1q_1^x + C_2q_2^x + \ldots + C_nq_n^x$, where the C_n are arbitrary constants.

(ii) Let us suppose that one root, q_1, is complex. Therefore, if the coefficients b_α are real, also $q_2 \equiv \overline{q_1}$ is a solution of the equation. The general solution of (A.1) is now $u(x) = C_1q_1^x + C_2\overline{q_1}^x + \ldots + C_nq_n^x$.

(iii) The third situation appears, for instance, when all the roots but two are real and distinct while $q_1 = q_2$. In this case the general solution is $u(x) = C_1q_1^x + C_2xq_1^x + \ldots + C_nq_n^x$.

If the equation is not homogeneous, we have to add to the general solution of (A.1) a particular solution of the complete equation. As we can see the theory (and practice) of difference equations strongly resembles the theory of ordinary differential equations.

A particular solution of the complete equation

$$L(E)u(x) = \Phi(x) \tag{A.2}$$

can be very easily found for particular examples of the function $\Phi(x)$. For instance, if $\Phi(x) = a^x$, then a particular solution is $u_p(x) = [1/L(a)]a^x$ whenever $L(a) \neq 0$. Of course, due to the linearity of the operator L, it is very easy to find a particular solution even if we have $\Phi(x) = \cos(\alpha x)$ or $\Phi(x) = \sin(\alpha x)$. Moreover, following common sense, when $\Phi(x)$ is a polynomial of degree m, $P_m(x)$, we look for a particular solution of (A.2) as a polynomial of the same degree, $Q_m(x)$. Still, if $\Phi(x) = \beta^xP_m(x)$, then we have $u_p(x) = \beta^xQ_m(x)$. And yet, for $\Phi(x) = \beta^x\sin(\alpha x)$ or $\Phi(x) = \beta^x\cos(\alpha x)$, we search for a solution of the form $u_p(x) = \beta^x(A\cos(\alpha x) + B\sin(\alpha x))$. More information, techniques, and tricks can be found in ref. 8 and other textbooks of numerical analysis.

APPENDIX B. STANDARD INVERSE CALCULUS

This appendix is devoted to giving some information about the standard inverse calculus of variation, information translated in Section 4 into our nonstandard language. We do not plan to give a general overview of this subject, which is widely discussed in the literature (see, e.g., refs. 9–11 and references therein). Let us consider the following system of first-order differential equations:

$$
\begin{cases}
\dot{x}_1 = \phi_1(x_1, x_2, \ldots, x_n, t) \\
\dot{x}_2 = \phi_2(x_1, x_2, \ldots, x_n, t) \\
\ldots \\
\ldots \\
\dot{x}_n = \phi_n(x_1, x_2, \ldots, x_n, t)
\end{cases}
\tag{B.1}
$$

It is well known that any differential equation of order n can be rewritten in this form by simply introducing new variables. It is discussed in the literature that not all the systems like (B.1) can be obtained by a quadratic Lagrangian.[11] Let us therefore consider a first-order Lagrangian for such a system,

$$
L(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, t) = \sum_{\alpha=1}^{n} \dot{x}_\alpha f_\alpha(x_1, \ldots, x_n, t) + f_0(x_1, \ldots, x_n, t) \tag{B.2}
$$

where all the functions f_β do not depend on $\dot{x}_j$. Using the standard Euler–Lagrange equations (3.3), we obtain a certain number of constraints for the functions f_β. These are necessary conditions for the Lagrangian in (B.2) to give back the set (B.1). Defining

$$
M_{\alpha\beta} \equiv \frac{\partial f_\alpha}{\partial x_\beta} - \frac{\partial f_\beta}{\partial x_\alpha}, \qquad F_\alpha \equiv \frac{\partial f_0}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial t} \tag{B.3}
$$

it is easy to see that the following set of n equations must be satisfied:

$$
\sum_{\beta=1}^{n} M_{\alpha\beta}\phi_\beta = F_\alpha, \qquad \alpha = 1, 2, \ldots, n \tag{B.4}
$$

A first interesting consequence of this constraint is that, if n is an odd integer, then there does not exist any first-order Lagrangian for the system (B.1). The reason is simply that the skew-symmetric matrix M is necessarily singular if the number of the rows is odd.

The situation is different if n is an even integer, since in this case in general we have $\det M \neq 0$. In ref. 9 it is discussed that, once f_0 is fixed, there surely exists a solution f_α of the system (B.4). In this paper we have focused on second-order differential equations since they have a particular relevance in classical mechanics. We consider therefore the following particular form of the system (B.1):

$$
\begin{cases}
\dot{x}_1 = \phi_1(x_1, x_2, t) \\
\dot{x}_2 = \phi_2(x_1, x_2, t)
\end{cases}
\tag{B.5}
$$

We list here some particular situations in which the functions f_0, f_1, f_2 can be easily found. It is very simple to verify this result and we will not do it here. We assume that all the quantities below are assumed to be well defined.

1. $\phi_1(x_1, x_2, t) = \phi_1(x_2), \qquad \phi_2(x_1, x_2, t) = \phi_2(x_1) \Rightarrow$

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = \dot{x}_1 x_2 + \int \phi_2(x_1)\, dx_1 - \int \phi_1(x_2)\, dx_2$$

2. $\phi_1(x_1, x_2, t) = \phi_1(x_1), \qquad \phi_2(x_1, x_2, t) = \phi_2(x_2) \Rightarrow$

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = \dot{x}_1 \frac{1}{\phi_1(x_1)} \int \frac{dx_2}{\phi_2(x_2)} + \int \frac{dx_1}{\phi_1(x_1)} - \int \frac{dx_2}{\phi_2(x_2)}$$

3. $\phi_1(x_1, x_2, t) = x_2, \qquad \phi_2(x_1, x_2, t) = \phi_2(x_1) \Rightarrow$

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = \dot{x}_1 x_2 + \int \phi_2(x_1)\, dx_1 - \frac{x_2^2}{2}$$

4. $\phi_1(x_1, x_2, t) = x_2, \qquad \phi_2(x_1, x_2, t) = \phi_2(x_2) \Rightarrow$

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = \dot{x}_1 \int \frac{dx_2}{\phi_2(x_2)} + \int \frac{x_2\, dx_2}{\phi_2(x_2)} - x_1$$

5. $\phi_1(x_1, x_2, t) = x_2, \qquad \phi_2(x_1, x_2, t) = \Phi(x_1)\, \Psi(x_2) \Rightarrow$

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = \frac{\dot{x}_2 x_1}{\Psi(x_2)} - \int \phi(x_1)\, dx_1 + \int \frac{x_2 dx_2}{\Psi(x_2)}$$

We make some brief remarks before concluding:

(i) Situation 2 describes two decoupled equations.

(ii) Situations 3–5 describe three different examples of classical equations of motion, so that they have a direct physical interpretation.

(iii) Whenever the functions ϕ_α do not depend explicitly on t, even the functions f_α may be chosen to be time independent.

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